Lie Superalgebras and the Multiplet Structure of the Genetic Code II: Branching Schemes

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Abstract

Continuing our attempt to explain the degeneracy of the genetic code using basic classical Lie superalgebras, we present the branching schemes for the typical codon representations (typical 64-dimensional irreducible representations) of basic classical Lie superalgebras and find three schemes that do reproduce the degeneracies of the standard code, based on the orthosymplectic algebra $\mathfrak{osp}(5|2)$ and differing only in details of the symmetry breaking pattern during the last step.

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1 Introduction

In the context of the project proposed by Hornos & Hornos [1] which aims at explaining the degeneracy of the genetic code as the result of a symmetry breaking process, we have carried out a systematic analysis of the possibility to implement this idea by starting out from a typical codon representation (typical 64-dimensional irreducible representation) of a basic classical Lie superalgebra, rather than a codon representation (64-dimensional irreducible representation) of an ordinary simple Lie algebra. The investigation of such an algebraic approach to the genetic code using alternative concepts of symmetry such as supersymmetry, where ordinary Lie algebras are replaced by Lie superalgebras, has already been suggested in the original paper [1] – except for the restriction to basic classical Lie superalgebras (a particular class of simple Lie superalgebras) and to typical representations (a particular class of irreducible representations): only under this restriction, which is of a technical nature, does there exist a sufficiently well developed mathematical theory, due to Kac [2, 3], to allow for the kind of analysis that is necessary to carry out such a program. As a first step, we have in a previous paper [4] presented a complete classification of all typical codon representations of basic classical Lie superalgebras: there are altogether 18 such representations involving 12 different Lie superalgebras. Our goal in the present paper is to analyze all possible branching schemes that can be obtained from these representations with regard to their capability of reproducing the degeneracy of the genetic code, following the strategy used in ref. [1] and explained in detail in ref. [5], but with one essential restriction: supersymmetry will be broken right away, in the very first step.

To motivate this assumption, note that the distribution of multiplets found in the genetic code today does not appear to correspond to the kind of scheme one would expect from the representation theory of Lie superalgebras. Thus if some kind of supersymmetry has been present at the very beginning of the evolution of the genetic code, it must have been broken. Moreover, it does not seem plausible to us that this breaking should have occurred only in the last step of the process, where the phenomenon of "freezing" would have been able to prevent a complete breakdown (see ref. [5] for more details). But if supersymmetry has been broken before, then there is mathematically no loss of generality in assuming that it has been broken in the very first step, because as soon as we may exclude freezing, symmetry breaking through chains of subalgebras that differ only in the order in which the successive steps are performed (such as $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_1 \cap \mathfrak{g}_2$ and $\mathfrak{g} \supset \mathfrak{g}_2 \supset \mathfrak{g}_1 \cap \mathfrak{g}_2$) will lead to the same end result.

2 The first step: Breaking the supersymmetry

With the above picture in mind and using the fact that among the semisimple ordinary Lie algebras which are subalgebras of a given basic classical Lie superalgebra $\mathfrak{g}$, there is a unique maximal one, namely the semisimple part $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$ of its even part $\mathfrak{g}_{\bar{0}}$, our task for the first step of the symmetry breaking process is to compute, for each of the 18 codon representations of the 12 basic classical Lie superalgebras found in ref. [4], its branching into irreducible representations under restriction from $\mathfrak{g}$ to $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$. There are two different methods for doing this. One consists in computing all weight vectors that result from the action of products of generators associated with the negative odd roots on the highest weight vector, where every negative odd root appears at most once in such a product: these are the candidates for highest weight vectors of irreducible representations of $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$ that appear in the direct decomposition of the original codon representation of $\mathfrak{g}$. The problem is to decide which of these representations really appear, and with what multiplicity. Although there is an explicit formula for calculating such multiplicities, due to Kac and Kostant, the procedure involves a summation over the Weyl group and is cumbersome to apply in practice. Therefore, we shall, following common usage, adopt the other method, which is based on the use of Young superdiagrams – a generalization of the usual Young diagrams from ordinary Lie algebras to Lie superalgebras.

In order to understand how this technique works, it is useful to recall how Young diagrams arise in the representation theory of ordinary simple Lie algebras. Given a simple Lie algebra $\mathfrak{g}_0$, consider the first fundamental representation of $\mathfrak{g}_0$, i.e., the irreducible representation of $\mathfrak{g}_0$ with highest weight equal to the first fundamental weight, denoted in what follows by D. Alternatively, we may characterize D as the lowest-dimensional (non-trivial) irreducible representation of $\mathfrak{g}_0$: for the matrix Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$, it is simply the n-dimensional defining representation. The basic idea is now to look at all tensor powers $D^{\otimes p}$ of D and reduce them into their irreducible constituents. This reduction is achieved by considering symmetric tensors, antisymmetric tensors and, more generally, tensors of mixed symmetry type. In fact, permutation of the factors induces a representation of the symmetric group S_p on the representation space of $D^{\otimes p}$ and this action of S_p commutes with that of $\mathfrak{g}_0$, so that both actions can be simultaneously decomposed into irreducible constituents. More precisely, this is achieved by combining them into a "joint action"[1] and then performing a decomposition into irreducible constituents in the usual sense: each of these has the property that its multiplicity as a representation of S_p equals its

[1]The concept of "joint action" used here can be formulated in mathematically rigorous terms by introducing the connected, simply connected, simple Lie group G_0 corresponding to $\mathfrak{g}_0$ and considering D and $D^{\otimes p}$ as representations of G_0; then the joint action of S_p and $\mathfrak{g}_0$ corresponds to a representation of the direct product $S_p \times G_0$.

dimension as a representation of $\mathfrak{g}_0$ and its multiplicity as a representation of $\mathfrak{g}_0$ equals its dimension as a representation of S_p. The usefulness of this approach stems from an important theorem of Weyl which states that any irreducible representation of the classical Lie algebras $\mathfrak{sl}(n)$ and $\mathfrak{sp}(n)$, as well as any tensorial irreducible representation of the classical Lie algebras $\mathfrak{so}(n)$, can be obtained in this way.[2] Therefore, a Young diagram of p boxes, which originally stands for an irreducible representation of the symmetric group S_p, also determines an irreducible representation of $\mathfrak{g}_0$ contained in $D^{\otimes p}$. In the case of $\mathfrak{sl}(n)$, the latter is simply obtained by considering tensors of a specific symmetry type, given by the projection operator of symmetrizing along the rows and antisymmetrizing along the columns of the corresponding standard Young tableau [6, 7], whereas in the case of $\mathfrak{sp}(n)$ and $\mathfrak{so}(n)$, the existence of invariant bilinear forms for D (antisymmetric for $\mathfrak{sp}(n)$ and symmetric for $\mathfrak{so}(n)$) implies that this operation alone is not sufficient to produce an irreducible representation: here, a given Young diagram stands for tensors of the corresponding symmetry type which in addition are totally traceless with respect to the pertinent bilinear form, that is, traceless in all indices in which they are antisymmetric in the case of $\mathfrak{sp}(n)$ and traceless in all indices in which they are symmetric in the case of $\mathfrak{so}(n)$.

The rules for constructing Young tableaux and diagrams can be extended in such a way as to also cover spinorial representations of $\mathfrak{so}(n)$. To this end, one must include the spinor representation(s), i.e., the standard spinor representation S of highest weight $(0,\ldots,0,1)$ if n is odd and the two chiral spinor representations S^+ and S^-, of highest weight $(0,\ldots,0,1,0)$ and $(0,\ldots,0,0,1)$, respectively, if n is even: this turns out to be sufficient because according to a modified form of Weyl's theorem, an arbitrary irreducible representation of $\mathfrak{so}(n)$ can be obtained as a subrepresentation of the representation $D^{\otimes p} \otimes S$ if n is odd and of one of the two representations $D^{\otimes p} \otimes S^+$ or $D^{\otimes p} \otimes S^-$ if n is even, for adequate p. Therefore, it is convenient to introduce generalized Young tableaux and diagrams containing "spinor" or "half" boxes, one at the beginning of each row[3] and characterized by inserting the letter "s" into each of them, as well as a possible "negative" last row instead of the usual "positive" one when n is even, thus allowing to distinguish between the two chiralities for the spinors. For a summary of the conventions that we shall follow, the reader is referred to the Appendix of ref. [8].

An important point to be noticed is that although different (generalized) Young diagrams correspond to different irreducible representations of the permu-

[2] An irreducible representation of $\mathfrak{so}(n)$ of highest weight $(l_1,\ldots,l_{r-1},l_r)$ is tensorial, or non-spinorial, if l_r is even for $n=2r+1$ odd (B-series) and if $l_{r-1}+l_r$ is even for $n=2r$ even (D-series).

[3] The property of having only one spinor box per row reflects the fact that the spinor representation(s) appear only once in the tensor product, so that in particular, there is no problem with symmetrization or antisymmetrization of spinor indices.

tation group S_p, they may very well describe the same irreducible representation of $\mathfrak{g}_0$: thus the characterization of irreducible representations of $\mathfrak{g}_0$ by (generalized) Young diagrams is ambiguous. In order to remove this ambiguity, one introduces *modification rules* which allow to reduce every (generalized) Young diagram to its *standard* form, as explained, for instance, in [9]: this is done in such a way that every irreducible representation corresponds to precisely one standard (generalized) Young diagram.

The technique of (generalized) Young tableaux and Young diagrams for characterizing irreducible representations has been extended from the classical simple Lie algebras to the special linear and orthosymplectic Lie superalgebras, giving rise to *Young supertableaux* and *Young superdiagrams*, which we shall distinguish from their non-supersymmetric counterparts by the insertion of a diagonal line across each box. They describe typical representations as well as atypical ones. As in the non-supersymmetric case, several different Young superdiagrams may provide the same irreducible representation, and modification rules are needed to remove the ambiguity: they serve to reduce a Young superdiagram to its *standard* form. For an atypical representation, this procedure is still not unambiguous, leading to different standard Young superdiagrams describing the same representation, whereas for a typical representation, the corresponding Young superdiagram can be constructed directly from its highest weight, and conversely, the Kac-Dynkin labels of the highest weight may be read off from the Young superdiagram. Note that fixing the highest weight includes fixing the Kac-Dynkin label l_s of the simple odd root, which for type I Lie superalgebras can take continuous values: the corresponding irreducible representation will in that case carry an additional continuous parameter. Its dimension and its branching rules under reduction from $\mathfrak{g}_0$ to $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$ will however not depend on the value of l_s which in [4] had remained unspecified, except for the constraints imposed by requiring typicality of the representation. Here, we shall make a choice for l_s that leads to the simplest possible Young superdiagram which is consistent with these constraints; this value, together with the resulting Young superdiagram, is specified in Tables 1-3.

For the special linear Lie superalgebras $\mathfrak{sl}(m \,|\, n)$, the procedure of constructing irreducible representations from Young superdiagrams is straightforward. The main difference from the case of the special linear Lie algebras $\mathfrak{sl}(n)$ is that the process of symmetrization and antisymmetrization involved in the definition of the Young idempotents that project onto a tensor of a specific symmetry type must now be understood in the appropriate supersymmetric or graded sense: symmetrization or antisymmetrization of two fermionic indices involves an extra minus sign to take into account the anticommuting character of these variables. This implies that there no longer exists an invariant totally antisymmetric tensor of top degree (invariant volume or ϵ-tensor), so the irreducible representations D and $\bar{D}$ become independent; therefore Young superdiagrams for $\mathfrak{sl}(m \,|\, n)$ are in

general composed of "undotted" and "dotted" boxes, as happens in the case of Young diagrams for $\mathfrak{gl}(n)$. For the applications needed in this paper, however, we shall find it sufficient to use the conventional type of Young superdiagram containing only "undotted" boxes.

The relation between such Young superdiagrams and Kac-Dynkin labels of irreducible representations for $\mathfrak{sl}(m\,|\,n)$ can be summarized as follows. First, recall that an ordinary Young diagram, characterized by a nonincreasing sequence $b_1 \geq \ldots \geq b_r$ of positive integers giving the lengths of its r rows[4], will be an allowed Young diagram for $\mathfrak{sl}(n)$ if and only if $r \leq n$; in this case it will describe an irreducible representation of $\mathfrak{sl}(n)$ with Dynkin labels $l_1, \ldots, l_{n-1}$ given by

$$l_i \;=\; b_i - b_{i+1} \qquad \text{for } i = 1, \ldots, n-1 \;, \tag{1}$$

Similarly, according to ref. [10], a Young superdiagram containing only "undotted" boxes, characterized by nonincreasing sequences $b_1 \geq \ldots \geq b_r$ and $c_1 \geq \ldots \geq c_s$ of positive integers giving the lengths of its r rows and s columns, respectively,[5], will be an allowed Young superdiagram for $\mathfrak{sl}(m\,|\,n)$ if and only if $b_{m+1} \leq n$; in this case it will describe an irreducible representation of $\mathfrak{sl}(m\,|\,n)$ whose Kac-Dynkin labels $l_1, \ldots, l_{m+n-1}$ can be found as follows. Define the reduced column lengths by

$$c_j' \;=\; (c_j - m)\,\theta(c_j - m) \;, \tag{2}$$

where θ is the step function; then

$$\begin{aligned} l_i \;&=\; b_i - b_{i+1} \qquad \text{for } i = 1, \ldots, m-1 \;, \\ l_m \;&=\; b_m + c_1' \;, \\ l_{m+j} \;&=\; c_j' - c_{j+1}' \qquad \text{for } j = 1, \ldots, n-1 \;. \end{aligned} \tag{3}$$

On the other hand, the branching rules under reduction from the Lie superalgebra $\mathfrak{sl}(m\,|\,n)$ to the semisimple part $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$ of its even subalgebra in terms of Young diagrams and superdiagrams can, according to ref. [11], be derived immediately from the corresponding branching rules under reduction from the ordinary Lie algebra $\mathfrak{sl}(m+n)$ under restriction to the same subalgebra $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n)$, which in turn are given in ref. [12], for a large class of examples. In fact all that needs to be done is to replace the Young diagram for the second summand $\mathfrak{sl}(n)$, which represents the odd sector of the representation space, by its transposed diagram, exchanging rows and columns. As an example, we show on the next page the decomposition of the Young superdiagram given by $r=2$, $s=2$ with $b_1 = 3 = c_1$, $b_2 = 2 = c_2$ and $b_3 = 1 = c_3$ which, according to eqns (2) and (3), corresponds to the typical codon representation of $\mathfrak{sl}(3\,|\,1)$, of highest weight $(1, 1, l_3)$ with $l_3 = 1$, as well as to the typical codon representation of

[4]It is to be understood that $b_r > 0$ but $b_i = 0$ if $i > r$.

[5]It is to be understood that $b_r > 0$ and $c_s > 0$ but $b_i = 0$ if $i > r$ and $c_j = 0$ if $j > s$.

$$
\begin{aligned}
\underset{\substack{\mathfrak{sl}(3\,|\,1)\\ \mathfrak{sl}(2\,|\,2)}}{\ydiagram{3,2,1}} \longrightarrow\;
& \underset{\substack{(1,1)\\ \text{illegal}}}{\left(\ydiagram{3,2,1}\,,1\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(1\,,\ydiagram{3,2,1}\right)}
+ 2\times \underset{\substack{\text{illegal}\\ (1-1)}}{\left(\ydiagram{2,1}\,,\ydiagram{2,1}\right)} \\
& + \underset{\substack{(1,2)\\ (1-1)}}{\left(\ydiagram{3,2}\,,\ydiagram{1}\right)}
+ \underset{\substack{(2,0)\\ \text{illegal}}}{\left(\ydiagram{3,1,1}\,,\ydiagram{1}\right)}
+ \underset{\substack{(0,1)\\ \text{illegal}}}{\left(\ydiagram{2,2,1}\,,\ydiagram{1}\right)} \\
& + \underset{\substack{\text{illegal}\\ (2-0)}}{\left(\ydiagram{3,1}\,,\ydiagram{1,1}\right)}
+ \underset{\substack{\text{illegal}\\ (0-0)}}{\left(\ydiagram{2,2}\,,\ydiagram{1,1}\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{2,1,1}\,,\ydiagram{1,1}\right)} \\
& + \underset{\substack{(2,1)\\ (2-2)}}{\left(\ydiagram{3,1}\,,\ydiagram{2}\right)}
+ \underset{\substack{(0,2)\\ (0-2)}}{\left(\ydiagram{2,2}\,,\ydiagram{2}\right)}
+ \underset{\substack{(1,0)\\ \text{illegal}}}{\left(\ydiagram{2,1,1}\,,\ydiagram{2}\right)} \\
& + \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{2,1}\,,\ydiagram{1,1,1}\right)}
+ \underset{\substack{\text{illegal}\\ (3-1)}}{\left(\ydiagram{3}\,,\ydiagram{2,1}\right)} \\
& + \underset{\substack{\text{illegal}\\ (1-1)}}{\left(\ydiagram{1}\,,\ydiagram{3,2}\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{1}\,,\ydiagram{3,1,1}\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{1}\,,\ydiagram{2,2,1}\right)} \\
& + \underset{\substack{\text{illegal}\\ (0-2)}}{\left(\ydiagram{1,1}\,,\ydiagram{3,1}\right)}
+ \underset{\substack{\text{illegal}\\ (0-0)}}{\left(\ydiagram{1,1}\,,\ydiagram{2,2}\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{1,1}\,,\ydiagram{2,1,1}\right)} \\
& + \underset{\substack{\text{illegal}\\ (2-2)}}{\left(\ydiagram{2}\,,\ydiagram{3,1}\right)}
+ \underset{\substack{\text{illegal}\\ (2-0)}}{\left(\ydiagram{2}\,,\ydiagram{2,2}\right)}
+ \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{2}\,,\ydiagram{2,1,1}\right)} \\
& + \underset{\substack{\text{illegal}\\ \text{illegal}}}{\left(\ydiagram{1,1,1}\,,\ydiagram{2,1}\right)}
+ \underset{\substack{(1,1)\\ (1-3)}}{\left(\ydiagram{2,1}\,,\ydiagram{3}\right)}
\end{aligned}
$$

$\mathfrak{sl}(2\,|\,2)$, of highest weight $(1, l_2, 1)$ with $l_2 = 3$. The highest weights with respect to $\mathfrak{sl}(3)$ and to $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ corresponding to the ordinary Young diagrams resulting from this decomposition are also exhibited and the "illegal" diagrams are identified: they are the ones that must be eliminated to comply with the prescription that Young diagrams for $\mathfrak{sl}(k)$ must not have more than k rows. In this way, we arrive at the branching schemes for the typical codon representations of $\mathfrak{sl}(m\,|\,n)$ given in Table 1 and in Table 2, since the remaining cases can be checked directly from the rules given in Table 1 of ref. [11].

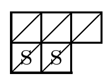
For the orthosymplectic Lie superalgebras $\mathfrak{osp}(M\,|\,N)$, where $M = 2m + 1$ or $M = 2m$ and $N = 2n$, the procedure is somewhat more complicated; it is described in ref. [8]. First of all, it must be noted that the construction and interpretation of Young superdiagrams for $\mathfrak{osp}(M\,|\,N)$, as compared to that for $\mathfrak{sl}(m\,|\,n)$, is subject to the same adjustments as that of ordinary Young diagrams for $\mathfrak{sp}(N)$ and $\mathfrak{so}(M)$, as compared to that for $\mathfrak{sl}(n)$: in particular, they may contain "spinor" or "half" boxes (referring, of course, only to the $\mathfrak{so}(M)$ part of the even subalgebra) which by convention will be located in the $(n+1)^{\mathrm{st}}$ row. We shall follow the notation of ref. [8], except that we shall continue to distinguish Young superdiagrams from their non-supersymmetric counterparts by the insertion of a diagonal line across each box, including the "spinor" or "half" boxes. The relation between the lengths $b_1 \geq \ldots \geq b_n$ of the first n rows and $c_1 \geq \ldots \geq c_m$ of the first m columns on the one hand and the Kac-Dynkin labels $l_1, \ldots, l_{n-1}, l_n, l_{n+1}, \ldots, l_{n+m}$ on the other hand is summarized in eqns (3.1), (3.4) and (3.5) of ref. [8]. The prescription for determining the branching rules under reduction from the Lie superalgebra $\mathfrak{osp}(M\,|\,N)$ to its even part $\mathfrak{sp}(N) \oplus \mathfrak{so}(M)$ has also been determined and is formally summarized in eqns (3.2), (3.3) and (3.6) of ref. [8]. The starting point is to dissect the given Young superdiagram into two ordinary Young diagrams: one for the $\mathfrak{sp}(N)$ part formed by the first n rows and one for the $\mathfrak{so}(M)$ part formed by the remaining rows, but reflected along the main diagonal. Together, they stand for the irreducible subrepresentation of the even subalgebra $\mathfrak{sp}(N) \oplus \mathfrak{so}(M)$ generated from the original highest weight vector by application of all even generators. It forms the ground floor of a building in which all the other irreducible subrepresentations of the even subalgebra are arranged in higher floors, each counted according to the minimum number of odd generators required to reach it from the ground floor. The procedure for determining which Young diagrams describe the irreducible subrepresentations that do appear in the higher floors is complicated, requiring the use of generalized Young diagrams for $\mathfrak{sp}(N)$ with negative boxes, as introduced in ref. [14], that must be multiplied to standard Young diagrams for $\mathfrak{so}(M)$, plus rules for eliminating Young diagrams resulting from this process that represent non-tracefree parts. A discussion of the general formulas presented in ref. [8] is not very instructive, so we prefer to just illustrate them by presenting two important examples:

the branching schemes for the typical codon representations of $\mathfrak{osp}(4\,|\,2)$ with highest weight $(\frac{7}{2},0,1)$ and of $\mathfrak{osp}(5\,|\,2)$ with highest weight $(\frac{5}{2},0,1)$.

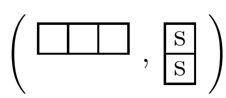
We begin by calculating the $\mathfrak{sp}(2) \oplus \mathfrak{so}(4)$ content of the typical codon representation of $\mathfrak{osp}(4\,|\,2)$ with highest weight $(\frac{7}{2},0,1)$. According to eqn (3.4) of ref. [8], the labels $b_1 \geq \ldots \geq b_n$ and $c_1 \geq \ldots \geq c_m$ of the corresponding Young superdiagram are given by

$$\begin{aligned} b_1 &= l_1^0 = l_1 - \tfrac{1}{2}(l_2 + l_3) = 3 \;, \\ c_1 &= n + \tfrac{1}{2}(l_3 + l_2) = \tfrac{3}{2} \;, \\ c_2 &= n + \tfrac{1}{2}(l_3 - l_2) = \tfrac{3}{2} \;, \end{aligned} \tag{4}$$

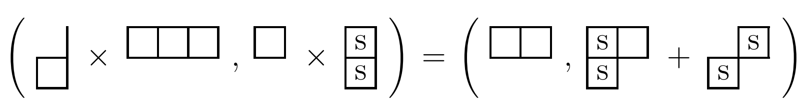
so the Young superdiagram has the form indicated in Table 3:



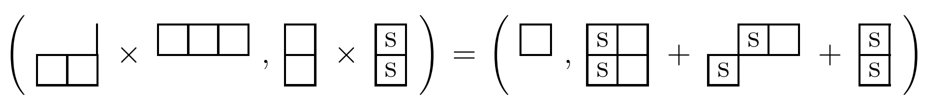
Therefore, the Young diagram for the even subalgebra $\mathfrak{sp}(2) \oplus \mathfrak{so}(4)$ is:



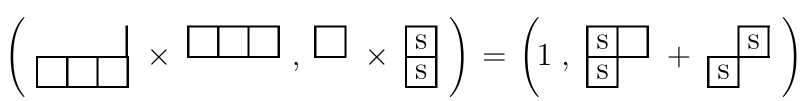
It describes the irreducible representation of highest weight $(3) - (0,1)$ which forms the ground floor. The irreducible representations on the following floors are computed graphically as follows:

1. floor: 

corresponding to the highest weights $(2) - (1,2)$ and $(2) - (1,0)$,

2. floor: 

corresponding to the highest weights $(1) - (0,3)$, $(1) - (2,1)$ and $(1) - (0,1)$,

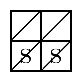
3. floor: 

corresponding to the highest weights $(0) - (1,2)$ and $(0) - (1,0)$. These are precisely the highest weights listed in Table 3 for this case.

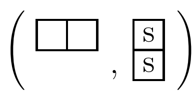
We proceed to calculate the $\mathfrak{sp}(2) \oplus \mathfrak{so}(5)$ content of the typical codon representation of $\mathfrak{osp}(5\,|\,2)$ with highest weight $(\frac{5}{2}, 0, 1)$. According to eqn (3.1) of ref. [8], the labels $b_1 \geq \ldots \geq b_n$ and $c_1 \geq \ldots \geq c_m$ of the corresponding Young superdiagram are given by

$$\begin{aligned} b_1 \;&=\; l_1^0 \;=\; l_1 - l_2 - \tfrac{1}{2} l_3 \;=\; 2 \, , \\ c_1 \;&=\; n + l_2 + \tfrac{1}{2} l_3 \;=\; \tfrac{3}{2} \, , \\ c_2 \;&=\; n + \tfrac{1}{2} l_3 \;=\; \tfrac{3}{2} \, , \end{aligned} \tag{5}$$

so the Young superdiagram has the form indicated in Table 3:

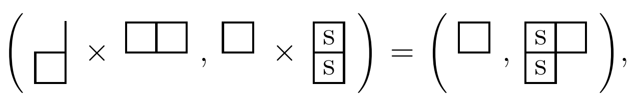


Therefore, the Young diagram for the even subalgebra $\mathfrak{sp}(2) \oplus \mathfrak{so}(5)$ is:



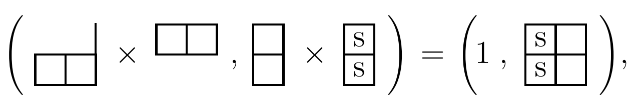
It describes the irreducible representation of highest weight $(2) - (0, 1)$ which forms the ground floor. The irreducible representations on the following floors are computed graphically as follows:

1. floor:



corresponding to the highest weight $(1) - (1, 1)$,

2. floor:



corresponding to the highest weight $(0) - (0, 3)$. Again, these are precisely the highest weights listed in Table 3 for this case.

Finally, it should be mentioned that we have omitted from Tables 1-3 some of the typical codon representations determined in [4] because their branching schemes are obvious from those that are listed. Examples are the typical codon representations of $\mathfrak{sl}(4\,|\,1)$ with highest weight $(0, 0, 1, l_4)$ and of $\mathfrak{sl}(2\,|\,2)$ with highest weight $(0, l_2, 3)$, which are complex conjugate to those with highest weight $(1, 0, 0, l_4)$ and $(3, l_2, 0)$, respectively, and which therefore exhibit the same branching schemes, in all phases, except for complex conjugation which however does not affect dimensions. Similarly, it is known that the branching rules of typical representations of the Lie superalgebra $\mathfrak{osp}(4\,|\,2, \alpha)$ upon reduction to its

even part do not depend on α [15], so that we may without loss of generality put $\alpha = 1$. Moreover, our calculations have shown that the three typical representations with highest weight $(5,0,0)$, $(\frac{7}{2},3,0)$ and $(\frac{7}{2},0,3)$, as well as the three typical representations with highest weight $(3,1,1)$, $(\frac{7}{2},1,0)$ and $(\frac{7}{2},0,1)$, although inequivalent, have the same branching rules under this reduction, so we have listed only one of each.

3 The Search for Surviving Chains

In the preceding section, we have described in some detail the arguments that are needed to analyze the first step of the symmetry breaking process through chains of subalgebras, during which the original supersymmetry is removed. All further steps involve only ordinary Lie algebras and are carried out according to the strategy already used in [1] and explained in detail in [5]; see also [16]. Briefly, the main criterion for excluding a given chain without having to analyze all of its ramifications is the occurence of one of the following situations:

- Total pairing: all multiplets come in pairs of equal or complex conjugate representations. No further breaking is able to remove this feature, excluding the possibility to produce multiplets with odd multiplicity, that is, the 3 sextets, 5 quartets and 9 doublets found in the genetic code.

- More than 2 singlets. No further breaking is able to reduce the number of singlets, excluding the possibility to produce no more than the 2 singlets found in the genetic code.

- More than 4 odd-dimensional multiplets. No further breaking is able to reduce the number of odd-dimensional multiplets, excluding the possibility to produce no more than the 2 triplets and 2 singlets found in the genetic code.

In what follows, we list the chains that can be excluded by one of these arguments, together with the relevant information on the distribution of multiplets obtained after the last step.

- $A(2\,|\,0) = \mathfrak{sl}(3\,|\,1)$:

  Total pairing.

- $A(3\,|\,0) = \mathfrak{sl}(4\,|\,1)$:

  Continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

  - $A(3\,|\,0) \supset A_3 \supset A_2$: 10 triplets and 6 singlets.
  - $A(3\,|\,0) \supset A_3 \supset C_2 \supset A_1 \oplus A_1$: 4 triplets and 4 singlets.

- $A(3\,|\,0) \supset A_3 \supset C_2 \supset A_1$:
  2 septets, 2 quintets, 2 triplets and 2 singlets.
- $A(3\,|\,0) \supset A_3 \supset A_1 \oplus A_1$: 2 nonets, 4 triplets and 2 singlets.

- $A(5\,|\,0) = \mathfrak{sl}(6\,|\,1)$:

  Continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

  - $A(5\,|\,0) \supset A_5 \supset A_4$: 4 quintets and 4 singlets, as well as total pairing.
  - $A(5\,|\,0) \supset A_5 \supset A_3$: Total pairing.
  - $A(5\,|\,0) \supset A_5 \supset C_3$: 4 singlets.
  - $A(5\,|\,0) \supset A_5 \supset A_2$: Total pairing.
  - $A(5\,|\,0) \supset A_5 \supset A_1 \oplus A_3$: 4 singlets.
  - $A(5\,|\,0) \supset A_5 \supset A_2 \oplus A_2$: 4 nonets, 8 triplets and 4 singlets.
  - $A(5\,|\,0) \supset A_5 \supset A_1 \oplus A_2 \supset A_1 \oplus A_1^{(1)}$, where $A_2 \supset A_1^{(1)}$ corresponds to $\mathfrak{su}(3) \supset \mathfrak{su}(2)$: 4 triplets and 4 singlets.
  - $A(5\,|\,0) \supset A_5 \supset A_1 \oplus A_2 \supset A_1 \oplus A_1^{(2)}$, where $A_2 \supset A_1^{(2)}$ corresponds to $\mathfrak{su}(3) \supset \mathfrak{so}(3)$: 2 nonets, 2 quintets and 4 singlets.

- $A(1\,|\,1)^c = \mathfrak{sl}(2\,|\,2)$, highest weight $(1, l_2, 1)$: [6]

  Too many odd-dimensional multiplets.

- $A(2\,|\,1) = \mathfrak{sl}(3\,|\,2)$:

  Continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

  - $A(2\,|\,1) \supset A_2 \oplus A_1 \supset A_1 \oplus A_1^{(1)}$, where $A_2 \supset A_1^{(1)}$ corresponds to $\mathfrak{su}(3) \supset \mathfrak{su}(2)$: 4 triplets and 4 singlets.
  - $A(2\,|\,1) \supset A_2 \oplus A_1 \supset A_1 \oplus A_1^{(2)}$, where $A_2 \supset A_1^{(2)}$ corresponds to $\mathfrak{su}(3) \supset \mathfrak{so}(3)$: 2 nonets, 2 quintets and 4 singlets.

- $C(3) = \mathfrak{osp}(2\,|\,4)$:

  Continuing the symmetry breaking process, we obtain the following chains, all of which can be excluded:

  - $C(3) \supset C_2 \supset A_1 \oplus A_1$: 4 triplets and 4 singlets.
  - $C(3) \supset C_2 \supset A_1$:
    2 septets, 2 quintets, 2 triplets and 2 singlets.

[6]The superscript $.^c$ stands for "central extension".

- $C(4) = \mathfrak{osp}(2\,|\,6)$:
  Too many singlets.

In the terminology of ref. [5], we are thus left with six basic classical Lie superalgebras whose codon representations, up to the end of the first phase of the symmetry breaking process, produce surviving chains: their remaining symmetry is described by a direct sum of $\mathfrak{sl}(2)$-algebras.

Finally, we must pass to the second phase of the symmetry breaking process, during which some of the $\mathfrak{sl}(2)$-algebras are broken. There are two ways of doing this, depending on whether one uses the operator L_z or the operator L_z^2 as the symmetry breaking term in the model Hamiltonian; we shall in what follows refer to these two possibilities as "strong" breaking and "soft" breaking, respectively. However, only the first of them corresponds to a genuine symmetry breaking at the level of Lie algebras, namely from the Lie algebra $\mathfrak{sl}(2)$ to its Cartan subalgebra. A natural interpretation of both possibilities as a legitimate symmetry breaking requires passing from the complex Lie algebra $\mathfrak{sl}(2)$ to its compact real form $\mathfrak{su}(2)$ and from there to the corresponding connected, simply connected Lie group $SU(2)$, which all have the same representation theory: then as has been observed in ref. [17], we may break the symmetry under the (connected) group $SU(2)$ in two different ways: a) down to its maximal connected subgroup $U(1) \cong SO(2)$ (strong breaking) or b) down to its maximal (non-connected) subgroup $\mathbb{Z}_2 \times U(1) \cong O(2)$ (soft breaking). The effect on a multiplet of dimension $2s+1$, corresponding to an irreducible representation of $SU(2)$ (or $\mathfrak{su}(2)$ or $\mathfrak{sl}(2)$) of spin s and highest weight $2s$, is to break it a) strongly into $2s+1$ singlets, corresponding to the different eigenvalues of the operator L_z, or b) softly into

- s doublets and one singlet if s is integer, or
- s doublets if s is half-integer,

corresponding to the different eigenvalues of the operator L_z^2.

The main complication in this second phase of the symmetry breaking process arises from the necessity to take into account the possibility of (partially) "freezing" the symmetry breakdown in the last step; for more details, see the discussion in ref. [5].

As an immediate consequence of the previous discussion, we see that the chain resulting from the codon representation of $\mathfrak{sl}(2\,|\,1)$ can be excluded: all multiplets are of dimension > 6 so that further symmetry breaking is needed (i.e., no freezing is allowed), but the remaining symmetry algebra being a single copy of $\mathfrak{sl}(2)$, any further breaking will produce only singlets or doublets.

The most stringent criterion for a chain to be surviving during the second phase of the symmetry breaking process comes from the requirement of producing the correct number of sextets (3) and triplets (2): it demands, among other things, that the number

$$d_3 \ = \ \begin{array}{c} \text{sum of the dimensions of all multiplets} \\ \text{whose dimension is a multiple of 3} \end{array}$$

which during this phase cannot decrease, must always remain ≥ 24. As an example, note that this condition immediately eliminates the codon representation of $\mathfrak{osp}(3\,|\,2)$, for which $d_3 = 18$, according to Table 3. The remaining cases must be handled case by case, as follows.

- $A(1\,|\,1)^c = \mathfrak{sl}(2\,|\,2)$, highest weight $(3, l_2, 0)$. [6]

  Up to the end of the first phase, we have a unique chain:

  $$\mathfrak{sl}(2\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

  The corresponding distribution of multiplets can be read off from Table 2; there are altogether 10 multiplets, with $d_3 = 30$. However, among the four multiplets whose dimension is a multiple of 3, we have one multiplet of dimension 6, namely $(5) - (0)$, which cannot break into triplets, one multiplet of dimension 12, namely $(3) - (2)$, which can either break into four triplets or else will produce no triplets at all, and finally two identical multiplets of dimension 6, namely $(2) - (1)$, which together can also either break into four triplets or else produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

- $B(1\,|\,2) = \mathfrak{osp}(3\,|\,4)$, highest weight $(0, \frac{5}{2}, 3)$.

  Up to the end of the first phase, we have the following chains.

  1. $\mathfrak{osp}(3\,|\,4) \supset \mathfrak{sp}(4) \oplus \mathfrak{so}(3) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets can be read off from Table 4; there are altogether 8 multiplets, with $d_3 = 24$. However, the two multiplets whose dimension is a multiple of 3, namely $(1)-(0)-(5)$ and $(0) - (1) - (5)$, both of dimension 12, cannot break into triplets, so this chain may be discarded.
  2. $\mathfrak{osp}(3\,|\,4) \supset \mathfrak{sp}(4) \oplus \mathfrak{so}(3) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets is identical with that shown in Table 3, since no further branching occurs in the second reduction; there are altogether 5 multiplets, with $d_3 = 24$. However, the unique multiplet whose dimension is a multiple of 3, namely $(3) - (5)$, of dimension 24, cannot break into triplets, so this chain may be discarded.

Continuing the first chain by diagonal breaking from three copies of $\mathfrak{sl}(2)$ to two gives rise to the following additional chain.

3. $\mathfrak{osp}(3\,|\,4) \supset \mathfrak{sp}(4) \oplus \mathfrak{so}(3) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \supset \mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$.

   The corresponding distribution of multiplets can be read off from Table 4; there are altogether 9 multiplets, with $d_3 = 36$. However, among the three multiplets whose dimension is a multiple of 3, we have two identical multiplets of dimension 12, namely $(1) - (5)$, which cannot break into triplets, and one other multiplet of dimension 12, namely $(2) - (3)$, which can either break into four triplets or else will produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

The other possibilities of diagonal breaking by contracting the first or second $\mathfrak{sl}(2)$ with the third can be ruled out because they lead to a total of 11 multiplets where the number d_3 has already dropped to 21, so there is no chance of producing the correct number of sextets and triplets.

- $B(2\,|\,1) = \mathfrak{osp}(5\,|\,2)$, highest weight $(\frac{5}{2}, 0, 1)$.

  Up to the end of the first phase, we have the following chains.

  1. $\mathfrak{osp}(5\,|\,2) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(5) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets can be read off from Table 5; there are altogether 10 multiplets, with $d_3 = 48$. Note also the symmetry of the distribution of multiplets under exchange of the second with the third $\mathfrak{sl}(2)$.

     In the first step, we must consider the following four options:

     1. breaking the first $\mathfrak{sl}(2)$ softly generates 12 multiplets with $d_3 = 36$,
     2. breaking the first $\mathfrak{sl}(2)$ strongly generates 18 multiplets with $d_3 = 36$,
     3. breaking the second $\mathfrak{sl}(2)$ softly generates 13 multiplets with $d_3 = 30$,
     4. breaking the second $\mathfrak{sl}(2)$ strongly generates precisely 21 multiplets with $d_3 = 30$.

     Note that the last option leads to an an interesting scheme that comes close to the genetic code but is slightly different, with 3 sextets, 5 quartets, 4 triplets, 5 doublets and 4 singlets. In the other three cases, the symmetry breaking process must proceed to the next stage, leading to the following options:

1.1 breaking the first $\mathfrak{sl}(2)$ down strongly generates 18 multiplets with $d_3 = 36$, so the symmetry breaking must continue and there can be no freezing at this stage, leading to the same situation as option 2 above,

1.2 breaking the second $\mathfrak{sl}(2)$ softly generates 15 multiplets with $d_3 = 18$,

1.3 breaking the second $\mathfrak{sl}(2)$ strongly generates 25 multiplets with $d_3 = 18$,

2.1 breaking the second $\mathfrak{sl}(2)$ softly generates 22 multiplets with $d_3 = 18$,

2.2 breaking the second $\mathfrak{sl}(2)$ strongly generates 35 multiplets with $d_3 = 18$,

3.1 breaking the first $\mathfrak{sl}(2)$ softly generates 15 multiplets with $d_3 = 18$,

3.2 breaking the first $\mathfrak{sl}(2)$ strongly generates 22 multiplets with $d_3 = 18$,

3.3 breaking the second $\mathfrak{sl}(2)$ down strongly generates precisely 21 multiplets with $d_3 = 30$, leading to the same situation as option 4 above,

3.4 breaking the third $\mathfrak{sl}(2)$ softly generates 16 multiplets with $d_3 = 12$,

3.5 breaking the third $\mathfrak{sl}(2)$ strongly generates 26 multiplets with $d_3 = 12$.

As before, options 1.2, 3.1 and 3.4 are excluded, whereas in the cases of options 1.3, 2.1, 2.2, 3.2 and 3.5, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. As it turns out, it is impossible to generate the correct number of sextets (3), triplets (2) and singlets (2). In the cases of options 1.3 and 3.5, we must break the multiplet of dimension 12 coming from the $(1-1-2)$ and can therefore generate at most 3 sextets or 2 sextets and 2 triplets. In the cases of options 2.1 and 3.2 (which without freezing would produce the same distribution of multiplets), there is no possibility of generating triplets. Finally, in the case of option 2.2, breaking or freezing any combination of the two doublets coming from the $(1-1-0)$, the two doublets coming from the $(0-3-0)$ and the three doublets coming from the $(2-1-0)$ will generate 14, 12, 10, 8, 6, 4 or no singlets, but not 2 singlets.

2. $\mathfrak{osp}(5\,|\,2) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(5) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

   The corresponding distribution of multiplets is easily obtained; there are altogether 7 multiplets, with $d_3 = 30$. However, among the three multiplets whose dimension is a multiple of 3, we have one multiplet of dimension 12, namely $(1) - (5)$, and one multiplet of dimension 6, namely $(0) - (5)$, both of which cannot break into triplets, and one other multiplet of dimension 12, namely $(2) - (3)$, which can either break into four triplets or else will produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

Continuing the first chain by diagonal breaking from three copies of $\mathfrak{sl}(2)$ to two gives rise to the following additional chain.

3. $\mathfrak{osp}(5\,|\,2) \supset \mathfrak{sp}(2) \oplus \mathfrak{so}(5) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \supset \mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$.

   The corresponding distribution of multiplets can be read off from Table 5; there are altogether 14 multiplets, with $d_3 = 33$.

   In the first step, we must consider the following four options:

   1. breaking the first $\mathfrak{sl}(2)$ softly generates precisely 21 multiplets with $d_3 = 18$,
   2. breaking the first $\mathfrak{sl}(2)$ strongly generates 35 multiplets with $d_3 = 18$,
   3. breaking the second $\mathfrak{sl}(2)$ softly generates 18 multiplets with $d_3 = 24$,
   4. breaking the second $\mathfrak{sl}(2)$ strongly generates 28 multiplets with $d_3 = 24$.

   Note that the first option leads to an an interesting scheme that comes close to the genetic code but is slightly different, with 2 sextets, 7 quartets, 2 triplets, 8 doublets and 2 singlets. In the cases of options 2 and 4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 9 must not be frozen, so we get at least 3 triplets and at least 6 odd-dimensional multiplets. Therefore, the only possibility of continuing the symmetry breaking process is case 3, leading to the following options:

   3.1 breaking the first $\mathfrak{sl}(2)$ softly generates 26 multiplets with $d_3 = 0$,

   3.2 breaking the first $\mathfrak{sl}(2)$ strongly generates 42 multiplets with $d_3 = 0$,

   3.3 breaking the second $\mathfrak{sl}(2)$ down strongly generates 28 multiplets with $d_3 = 24$.

In all three cases, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplet of dimension 8 must not be frozen and will break either into 2 quartets or into 4 doublets. In all three cases, we are able to reproduce the genetic code, provided the freezing is chosen appropriately, as shown in Tables 6-8.

The remaining possibility of diagonal breaking by contracting the second $\mathfrak{sl}(2)$ with the third can be ruled out because it leads to a total of 14 multiplets where the number d_3 has already dropped to 12, so there is no chance of producing the correct number of sextets and triplets.

- $D(2\,|\,1) = \mathfrak{osp}(4\,|\,2)$, highest weight $(5,0,0)$.

  Up to the end of the first phase, we have a unique chain:

  1. $\mathfrak{osp}(4\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets can be read off from Table 3; there are altogether 6 multiplets, with $d_3 = 42$. Note also the symmetry of the distribution of multiplets under exchange of the second with the third $\mathfrak{sl}(2)$. However, among the four multiplets whose dimension is a multiple of 3, we have one multiplet of dimension 6, namely $(5)-(0)-(0)$, which cannot break into triplets, and three multiplets of dimension 12, namely $(3)-(2)-(0)$, $(3)-(0)-(2)$ and $(2)-(1)-(1)$, each of which can either break into four triplets or else will produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

  Continuing this chain by diagonal breaking from three copies of $\mathfrak{sl}(2)$ to two gives rise to the following additional chains.

  2. $\mathfrak{osp}(4\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \supset \mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets is easily obtained; there are altogether 10 multiplets, with $d_3 = 36$. However, among the four multiplets whose dimension is a multiple of 3, we have one multiplet of dimension 12, namely $(5)-(1)$, and two identical multiplets of dimension 6, namely $(5)-(0)$, all of which cannot break into triplets, and one other multiplet of dimension 12, namely $(3)-(2)$, which can either break into four triplets or else will produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

3. $\mathfrak{osp}(4\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)_{23}$.

   The corresponding distribution of multiplets can be read off from Table 9; there are altogether 8 multiplets, with $d_3 = 57$.

   In the first step, we must consider the following four options:

   1. breaking the first $\mathfrak{sl}(2)$ softly generates 18 multiplets with $d_3 = 48$,
   2. breaking the first $\mathfrak{sl}(2)$ strongly generates 32 multiplets with $d_3 = 48$,
   3. breaking the second $\mathfrak{sl}(2)$ softly generates 12 multiplets with $d_3 = 18$,
   4. breaking the second $\mathfrak{sl}(2)$ strongly generates 16 multiplets with $d_3 = 18$.

   As before, options 3 and 4 are excluded, whereas in the case of option 2, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 and of dimension 5 must not be frozen, so we get at least 16 triplets and 5 singlets. Therefore, the only possibility of continuing the symmetry breaking process is case 1, leading to the following options:

   1.1 breaking the first $\mathfrak{sl}(2)$ down strongly generates 32 multiplets with $d_3 = 48$.

   1.2 breaking the second $\mathfrak{sl}(2)$ softly generates 27 multiplets with $d_3 = 0$,

   1.3 breaking the second $\mathfrak{sl}(2)$ strongly generates 35 multiplets with $d_3 = 0$.

   In all three cases, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, we already have 2 triplets and 2 singlets at the previous stage, and the requirement that no new triplets or singlets may be generated forces the large majority of the multiplets to be frozen. As it turns out, it is possible to generate the correct number of sextets (3), triplets (2) and singlets (2), but not of quartets (5) and doublets (9); we get at most 4 quartets and at least 11 doublets.

- $D(2\,|\,1) = \mathfrak{osp}(4\,|\,2)$, highest weight $(\frac{7}{2}, 0, 1)$.

  Up to the end of the first phase, we have a unique chain:

  1. $\mathfrak{osp}(4\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

     The corresponding distribution of multiplets can be read off from Table 3; there are altogether 8 multiplets, with $d_3 = 42$. Note also

the symmetry of the distribution of multiplets under exchange of the first with the third $\mathfrak{sl}(2)$.

In the first step, we must consider the following four options:

1. breaking the first $\mathfrak{sl}(2)$ softly generates 11 multiplets with $d_3 = 36$,
2. breaking the first $\mathfrak{sl}(2)$ strongly generates 18 multiplets with $d_3 = 36$,
3. breaking the second $\mathfrak{sl}(2)$ softly generates 9 multiplets with $d_3 = 30$,
4. breaking the second $\mathfrak{sl}(2)$ strongly generates 14 multiplets with $d_3 = 30$, but among them are 2 nonets, 4 triplets and 2 singlets.

In the first three cases, the symmetry breaking process must proceed to the next stage, leading to the following options:

1.1 breaking the first $\mathfrak{sl}(2)$ down strongly generates 18 multiplets with $d_3 = 36$, so the symmetry breaking must continue and there can be no freezing at this stage, leading to the same situation as option 2 above,

1.2 breaking the second $\mathfrak{sl}(2)$ softly generates 12 multiplets with $d_3 = 24$,

1.3 breaking the second $\mathfrak{sl}(2)$ strongly generates 19 multiplets with $d_3 = 24$, but among them are 4 triplets and 4 singlets,

1.4 breaking the third $\mathfrak{sl}(2)$ softly generates 15 multiplets with $d_3 = 12$,

1.5 breaking the third $\mathfrak{sl}(2)$ strongly generates 24 multiplets with $d_3 = 12$,

2.1 breaking the second $\mathfrak{sl}(2)$ softly generates 20 multiplets with $d_3 = 24$,

2.2 breaking the second $\mathfrak{sl}(2)$ strongly generates 30 multiplets with $d_3 = 24$,

2.3 breaking the third $\mathfrak{sl}(2)$ softly generates 24 multiplets with $d_3 = 12$,

2.4 breaking the third $\mathfrak{sl}(2)$ strongly generates 40 multiplets with $d_3 = 12$,

3.1 breaking the first $\mathfrak{sl}(2)$ softly generates 12 multiplets with $d_3 = 24$, leading to the same situation as option 1.2,

3.2 breaking the first $\mathfrak{sl}(2)$ strongly generates 20 multiplets with $d_3 = 24$, leading to the same situation as option 2.1,

3.3 breaking the second $\mathfrak{sl}(2)$ down strongly generates 14 multiplets with $d_3 = 30$, leading to the same situation as option 4 above.

As before, option 1.4 is excluded, whereas in the case of options 1.5, 2.2, 2.3 and 2.4, the symmetry breaking process must terminate, and we must take into account the possibility of freezing. However, the multiplets of dimension > 6 must not be frozen. In the cases of options 1.5 and 2.3, we do not get any triplets or singlets at all. In the case of option 2.4, we either do not get any triplets or singlets at all or else we get too many (at least 4). In the case of option 2.2, we are able to produce the correct number of sextets (3), triplets (2) and singlets (2), but there is no possibility to generating the correct number of quartets (5) and doublets (9): we can only get 2 quartets and 15 doublets. In the case of option 2.1, we already have 20 multiplets but no triplets and no singlets: their generation would require breaking at least two multiplets in the next step (one sextet and one doublet, for example), leading to at least 22 multiplets. We are thus left with a single surviving option for continuing the symmetry breaking process, namely $1.2 = 3.1$, which consists in breaking both the first and the second $\mathfrak{sl}(2)$ softly, generating 12 multiplets with $d_3 = 24$, giving rise to the following options:

a) breaking the first $\mathfrak{sl}(2)$ down strongly generates 20 multiplets with $d_3 = 24$, leading to the same situation as option 2.1 above,
b) breaking the second $\mathfrak{sl}(2)$ down strongly generates 19 multiplets with $d_3 = 24$, leading to the same situation as option 1.3 above,
c) breaking the third $\mathfrak{sl}(2)$ softly generates 16 multiplets with $d_3 = 0$,
d) breaking the third $\mathfrak{sl}(2)$ strongly generates 26 multiplets with $d_3 = 0$.

As before, option c) is excluded, whereas in the case of option d), we do not get any triplets or singlets at all.

Continuing this chain by diagonal breaking from three copies of $\mathfrak{sl}(2)$ to two gives rise to the following additional chain.

2. $\mathfrak{osp}(4\,|\,2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \supset \mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$.

   The corresponding distribution of multiplets is easily obtained; there are altogether 11 multiplets, with $d_3 = 24$. However, among the three multiplets whose dimension is a multiple of 3, we have one multiplet of dimension 12, namely $(3){-}(2)$, which can either break into four triplets or else will produce no triplets at all, and two identical multiplets of dimension 6, namely $(1) - (2)$, which together can also either break into four triplets or else produce no triplets at all. Thus there is no possibility to generate the two triplets found in the genetic code, so this chain may be discarded.

The remaining possibility of diagonal breaking by contracting the first $\mathfrak{sl}(2)$ with the third can be ruled out because it leads to a total of 14 multiplets among which there are 1 nonet, 2 quintets, 4 triplets and 1 singlet.

4 Conclusion

The main results of the analysis presented in ref. [4] and in the present paper, which in preliminary form were announced in [18] and [19], can be summarized as follows.

The idea of describing the degeneracies of the genetic code as the result of a symmetry breaking process through chains of subalgebras can be investigated systematically within the context of typical codon representations of basic classical Lie superalgebras, instead of ordinary codon representations of ordinary simple Lie algebras. The first result is negative: as before, there is no symmetry breaking pattern through chains of subalgebras capable of reproducing exactly the degeneracies of the genetic code. In other words, the phenomenon of "freezing" remains an essential part of the approach. The second result is positive and, as far as the uniqueness part is concerned, more stringent than its non-supersymmetric counterpart: admitting the possibility of "freezing" during the last step of the procedure, we find three schemes that do reproduce the degeneracies of the standard code, all based on the orthosymplectic algebra $\mathfrak{osp}(5|2)$ and differing only in the detailed form of the symmetry breaking pattern during the last step. The most natural scheme, shown in Tables 5 and 6, is the one that allows for a simple choice of Hamiltonian, in the sense used in ref. [1] and explained in more detail in ref. [5], namely the following:

$$\begin{aligned} H \;=\; & H_0 + \lambda\, C_2(\mathfrak{so}(5)) + \alpha_1 \boldsymbol{L}_1^2 + \alpha_2 \boldsymbol{L}_2^2 + \alpha_3 \boldsymbol{L}_3^2 + \alpha_{12} (\boldsymbol{L}_1 + \boldsymbol{L}_2)^2 \\ & + \beta_3 L_{3,z}^2 + \gamma_{12} \left((\boldsymbol{L}_1 + \boldsymbol{L}_2)^2 - 2 \right) (L_{1,z} + L_{2,z})^2 \;. \end{aligned} \tag{6}$$

The investigation of the resulting $\mathfrak{osp}(5\,|\,2)$ model for the genetic code is presently under way.

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Table 1: Branching of codon representations of type I Lie superalgebras in the first step $\mathfrak{g} \supset \mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$: Part 1

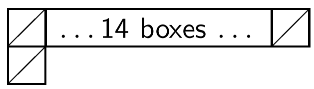
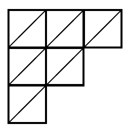
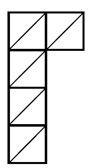
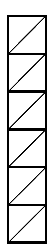
LSA $\mathfrak{g}$	Highest weight of $\mathfrak{g}$-multiplet	Young Superdiagram	Highest Weights of $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$-multiplets	d
$\mathfrak{sl}(2\,\vert\,1)$	$(15, l_2)$	$l_2 = 1$  	(16) $2 \times (15)$ (14)	17 2×16 15
$\mathfrak{sl}(3\,\vert\,1)$	$(1, 1, l_3)$	$l_3 = 1$ 	$(2, 1)$ $(1, 2)$ $2 \times (1, 1)$ $(2, 0)$ $(0, 2)$ $(1, 0)$ $(0, 1)$	15 15 2×8 6 6 3 3
$\mathfrak{sl}(4\,\vert\,1)$	$(1, 0, 0, l_4)$	$l_4 = 1$ 	$(1, 1, 0)$ $(1, 0, 1)$ $(2, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$ $2 \times (1, 0, 0)$ $(0, 0, 0)$	20 15 10 6 4 2×4 1
$\mathfrak{sl}(6\,\vert\,1)$	$(0, 0, 0, 0, 0, l_6)$	$l_6 = 1$ 	$(0, 0, 1, 0, 0)$ $(0, 1, 0, 0, 0)$ $(0, 0, 0, 1, 0)$ $(1, 0, 0, 0, 0)$ $(0, 0, 0, 0, 1)$ $2 \times (0, 0, 0, 0, 0)$	20 15 15 6 6 2×1

Table 2: Branching of codon representations of type I Lie superalgebras in the first step $\mathfrak{g} \supset \mathfrak{g}_0^{\mathrm{ss}}$: Part 2

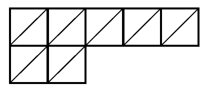
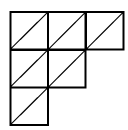
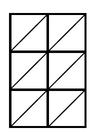
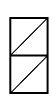
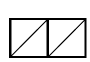
LSA $\mathfrak{g}$	Highest weight of $\mathfrak{g}$-multiplet	Young Superdiagram	Highest Weights of $\mathfrak{g}_0^{\mathrm{ss}}$-multiplets	d
$\mathfrak{sl}(2\,\vert\,2)$	$(3, l_2, 0)$	$l_2 = 2$ 	$(3) - (2)$ $2 \times ((4) - (1))$ $(5) - (0)$ $2 \times ((2) - (1))$ $3 \times ((3) - (0))$ $(1) - (0)$	12 2×10 6 2×6 3×4 2
	$(1, l_2, 1)$	$l_2 = 3$ 	$2 \times ((2) - (2))$ $(3) - (1)$ $(1) - (3)$ $4 \times ((1) - (1))$ $2 \times ((2) - (0))$ $2 \times ((0) - (2))$ $2 \times ((0) - (0))$	2×9 8 8 4×4 2×3 2×3 2×1
$\mathfrak{sl}(3\,\vert\,2)$	$(0, 0, l_3, 0)$	$l_3 = 2$ 	$(1,1) - (1)$ $(1,0) - (2)$ $(0,1) - (2)$ $(2,0) - (0)$ $(0,2) - (0)$ $(1,0) - (1)$ $(0,1) - (1)$ $(0,0) - (3)$ $2 \times ((0,0) - (0))$	16 9 9 6 6 6 6 4 2×1
$\mathfrak{osp}(2\,\vert\,4)$	$(l_1, 1, 0)$	$l_1 = 1$ 	$(1,1)$ $2 \times (2,0)$ $2 \times (0,1)$ $4 \times (1,0)$ $2 \times (0,0)$	16 2×10 2×5 4×4 2×1
$\mathfrak{osp}(2\,\vert\,6)$	$(l_1, 0, 0, 0)$	$l_1 = 3$ 	$(0,0,1)$ $2 \times (0,1,0)$ $3 \times (1,0,0)$ $4 \times (0,0,0)$	14 2×14 3×6 4×1

Table 3: Branching of codon representations of type II Lie superalgebras in the first step $\mathfrak{g} \supset \mathfrak{g}_{\bar{0}}$

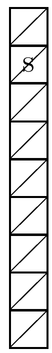
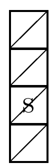
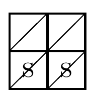
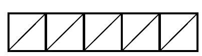
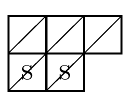
LSA $\mathfrak{g}$	Highest weight of $\mathfrak{g}$-multiplet	Young Superdiagram	Highest Weights of $\mathfrak{g}_{\bar{0}}$-multiplets	d
$\mathfrak{osp}(3\,\|\,2)$	$(\frac{17}{2}, 15)$		$(1) - (15)$ $(0) - (17)$ $(0) - (13)$	32 18 14
$\mathfrak{osp}(3\,\|\,4)$	$(0, \frac{5}{2}, 3)$		$(1,0) - (5)$ $(0,1) - (3)$ $(1,0) - (1)$ $(0,0) - (7)$ $(0,0) - (3)$	24 20 8 8 4
$\mathfrak{osp}(5\,\|\,2)$	$(\frac{5}{2}, 0, 1)$		$(1) - (1,1)$ $(0) - (0,3)$ $(2) - (0,1)$	32 20 12
$\mathfrak{osp}(4\,\|\,2)$	$(5, 0, 0)$		$(4) - (1) - (1)$ $(3) - (2) - (0)$ $(3) - (0) - (2)$ $(2) - (1) - (1)$ $(5) - (0) - (0)$ $(1) - (0) - (0)$	20 12 12 12 6 2
	$(\frac{7}{2}, 0, 1)$		$(2) - (1) - (2)$ $(1) - (2) - (1)$ $(3) - (0) - (1)$ $(1) - (0) - (3)$ $(2) - (1) - (0)$ $(0) - (1) - (2)$ $(1) - (0) - (1)$ $(0) - (1) - (0)$	18 12 8 8 6 6 4 2

Table 4: Branching of the codon representation of $\mathfrak{osp}(3\,|\,4)$ (first phase)

$\mathfrak{sp}(4) \oplus \mathfrak{so}(3)$		$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$	
Highest Weight	d	Highest Weight	d	Highest Weight	d
$(1,0)-(5)$	24	$(1)-(0)-(5)$	12	$(1)-(5)$	12
		$(0)-(1)-(5)$	12	$(1)-(5)$	12
$(0,1)-(3)$	20	$(1)-(1)-(3)$	16	$(2)-(3)$	12
				$(0)-(3)$	4
		$(0)-(0)-(3)$	4	$(0)-(3)$	4
$(1,0)-(1)$	8	$(1)-(0)-(1)$	4	$(1)-(1)$	4
		$(0)-(1)-(1)$	4	$(1)-(1)$	4
$(0,0)-(7)$	8	$(0)-(0)-(7)$	8	$(0)-(7)$	8
$(0,0)-(3)$	4	$(0)-(0)-(3)$	4	$(0)-(3)$	4
5 subspaces		8 subspaces		9 subspaces	

Table 5: Branching of the codon representation of $\mathfrak{osp}(5\,|\,2)$ (first phase)

$\mathfrak{sp}(2) \oplus \mathfrak{so}(5)$		$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$	
Highest Weight	d	Highest Weight	d	Highest Weight	d
$(1)-(1,1)$	32	$(1)-(2)-(1)$	12	$(3)-(1)$	8
				$(1)-(1)$	4
		$(1)-(1)-(2)$	12	$(2)-(2)$	9
				$(0)-(2)$	3
		$(1)-(1)-(0)$	4	$(2)-(0)$	3
				$(0)-(0)$	1
		$(1)-(0)-(1)$	4	$(1)-(1)$	4
$(0)-(0,3)$	20	$(0)-(2)-(1)$	6	$(2)-(1)$	6
		$(0)-(1)-(2)$	6	$(1)-(2)$	6
		$(0)-(3)-(0)$	4	$(3)-(0)$	4
		$(0)-(0)-(3)$	4	$(0)-(3)$	4
$(2)-(0,1)$	12	$(2)-(1)-(0)$	6	$(3)-(0)$	4
				$(1)-(0)$	2
		$(2)-(0)-(1)$	6	$(2)-(1)$	6
3 subspaces		10 subspaces		14 subspaces	

Table 6: Branching of the codon representation of $\mathfrak{osp}(5\,|\,2)$ (second phase): First option

$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$		$L^2_{3,z}$		$(L^2_{12,z}, L^2_{3,z})$	
$2s_1 - 2s_2 - 2s_3$	d	$2s_{12} - 2s_3$	d	$2s_{12} - 2m_3$	d	$2m_{12} - 2m_3$	d
$1-2-1$	12	$3-1$	8	$3-(\pm1)$	8	$(\pm3)-(\pm1)$	4
						$(\pm1)-(\pm1)$	4
		$1-1$	4	$1-(\pm1)$	4	$(\pm1)-(\pm1)$	4
$1-1-2$	12	$2-2$	9	$2-(\pm2)$	6	$(\pm2)-(\pm2)$	4
						$0-(\pm2)$	2
				$2-0$	3	$(\pm2)-0$	2
						$0-0$	1
		$0-2$	3	$0-(\pm2)$	2	$0-(\pm2)$	2
				$0-0$	1	$0-0$	1
$1-1-0$	4	$2-0$	3	$2-0$	3	$(\pm2)-0$	2
						$0-0$	1
		$0-0$	1	$0-0$	1	$0-0$	1
$1-0-1$	4	$1-1$	4	$1-(\pm1)$	4	$(\pm1)-(\pm1)$	4
$0-2-1$	6	$2-1$	6	$2-(\pm1)$	6	$(\pm2)-(\pm1)$	4
						$0-(\pm1)$	2
$0-1-2$	6	$1-2$	6	$1-(\pm2)$	4	$(\pm1)-(\pm2)$	4
				$1-0$	2	$(\pm1)-0$	2
$0-3-0$	4	$3-0$	4	$3-0$	4	$(\pm3)-0$	2
						$(\pm1)-0$	2
$0-0-3$	4	$0-3$	4	$0-(\pm3)$	2	$0-(\pm3)$	2
				$0-(\pm1)$	2	$0-(\pm1)$	2
$2-1-0$	6	$3-0$	4	$3-0$	4	$(\pm3)-0$	2
						$(\pm1)-0$	2
		$1-0$	2	$1-0$	2	$(\pm1)-0$	2
$2-0-1$	6	$2-1$	6	$2-(\pm1)$	6	$(\pm2)-(\pm1)$	4
						$0-(\pm1)$	2
10 subspaces		14 subspaces		18 subspaces		26 subspaces	

Table 7: Branching of the codon representation of $\mathfrak{osp}(5\,|\,2)$ (second phase): Second option

$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$		$L^2_{3,z}$		$(L_{12,z}, L^2_{3,z})$	
$2s_1 - 2s_2 - 2s_3$	d	$2s_{12} - 2s_3$	d	$2s_{12} - 2m_3$	d	$2m_{12} - 2m_3$	d
$1-2-1$	12	$3-1$	8	$3-(\pm1)$	8	$(+3)-(\pm1)$	2
						$(-3)-(\pm1)$	2
						$(+1)-(\pm1)$	2
						$(-1)-(\pm1)$	2
		$1-1$	4	$1-(\pm1)$	4	$(+1)-(\pm1)$	2
						$(-1)-(\pm1)$	2
$1-1-2$	12	$2-2$	9	$2-(\pm2)$	6	$(+2)-(\pm2)$	2
						$(-2)-(\pm2)$	2
						$0-(\pm2)$	2
				$2-0$	3	$(+2)-0$	1
						$(-2)-0$	1
						$0-0$	1
		$0-2$	3	$0-(\pm2)$	2	$0-(\pm2)$	2
				$0-0$	1	$0-0$	1
$1-1-0$	4	$2-0$	3	$2-0$	3	$(+2)-0$	1
						$(-2)-0$	1
						$0-0$	1
		$0-0$	1	$0-0$	1	$0-0$	1
$1-0-1$	4	$1-1$	4	$1-(\pm1)$	4	$(+1)-(\pm1)$	2
						$(-1)-(\pm1)$	2

Table 7 continued on next page

Table 7 continued from previous page							
$0-2-1$	6	$2-1$	6	$2-(\pm1)$	6	$(+2)-(\pm1)$	2
						$(-2)-(\pm1)$	2
						$0-(\pm1)$	2
$0-1-2$	6	$1-2$	6	$1-(\pm2)$	4	$(+1)-(\pm2)$	2
						$(-1)-(\pm2)$	2
				$1-0$	2	$(+1)-0$	1
						$(-1)-0$	1
$0-3-0$	4	$3-0$	4	$3-0$	4	$(+3)-0$	1
						$(-3)-0$	1
						$(+1)-0$	1
						$(-1)-0$	1
$0-0-3$	4	$0-3$	4	$0-(\pm3)$	2	$0-(\pm3)$	2
				$0-(\pm1)$	2	$0-(\pm1)$	2
$2-1-0$	6	$3-0$	4	$3-0$	4	$(+3)-0$	1
						$(-3)-0$	1
						$(+1)-0$	1
						$(-1)-0$	1
		$1-0$	2	$1-0$	2	$(+1)-0$	1
						$(-1)-0$	1
$2-0-1$	6	$2-1$	6	$2-(\pm1)$	6	$(+2)-(\pm1)$	2
						$(-2)-(\pm1)$	2
						$0-(\pm1)$	2
10 subspaces		14 subspaces		18 subspaces		42 subspaces	

Table 8: Branching of the codon representation of $\mathfrak{osp}(5\,|\,2)$ (second phase): Third option

$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2)_{12} \oplus \mathfrak{sl}(2)$		$L_{3,z}^2$		$L_{3,z}$	
$2s_1 - 2s_2 - 2s_3$	d	$2s_{12} - 2s_3$	d	$2s_{12} - 2m_3$	d	$2s_{12} - 2m_3$	d
$1-2-1$	12	$3-1$	8	$3-(\pm1)$	8	$3-(+1)$	4
						$3-(-1)$	4
		$1-1$	4	$1-(\pm1)$	4	$1-(+1)$	2
						$1-(-1)$	2
$1-1-2$	12	$2-2$	9	$2-(\pm2)$	6	$2-(+2)$	3
						$2-(-2)$	3
				$2-0$	3	$2-0$	3
		$0-2$	3	$0-(\pm2)$	2	$0-(+2)$	1
						$0-(-2)$	1
				$0-0$	1	$0-0$	1
$1-1-0$	4	$2-0$	3	$2-0$	3	$2-0$	3
		$0-0$	1	$0-0$	1	$0-0$	1
$1-0-1$	4	$1-1$	4	$1-(\pm1)$	4	$1-(+1)$	2
						$1-(-1)$	2
$0-2-1$	6	$2-1$	6	$2-(\pm1)$	6	$2-(+1)$	3
						$2-(-1)$	3
$0-1-2$	6	$1-2$	6	$1-(\pm2)$	4	$1-(+2)$	2
						$1-(-2)$	2
				$1-0$	2	$1-0$	2
$0-3-0$	4	$3-0$	4	$3-0$	4	$3-0$	4
$0-0-3$	4	$0-3$	4	$0-(\pm3)$	2	$0-(+3)$	1
						$0-(-3)$	1
				$0-(\pm1)$	2	$0-(+1)$	1
						$0-(-1)$	1
$2-1-0$	6	$3-0$	4	$3-0$	4	$3-0$	4
		$1-0$	2	$1-0$	2	$1-0$	2
$2-0-1$	6	$2-1$	6	$2-(\pm1)$	6	$2-(+1)$	3
						$2-(-1)$	3
10 subspaces		14 subspaces		18 subspaces		28 subspaces	

Table 9: Branching of the codon representation of $\mathfrak{osp}(4\,|\,2)$ with highest weight $(5,0,0)$ (first phase)

$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$		$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)_{23}$	
Highest Weight	d	Highest Weight	d
$(4)-(1)-(1)$	20	$(4)-(2)$	15
		$(4)-(0)$	5
$(3)-(2)-(0)$	12	$(3)-(2)$	12
$(3)-(0)-(2)$	12	$(3)-(2)$	12
$(2)-(1)-(1)$	12	$(2)-(2)$	9
		$(2)-(0)$	3
$(5)-(0)-(0)$	6	$(5)-(0)$	6
$(1)-(0)-(0)$	2	$(1)-(0)$	2
6 subspaces		8 subspaces	